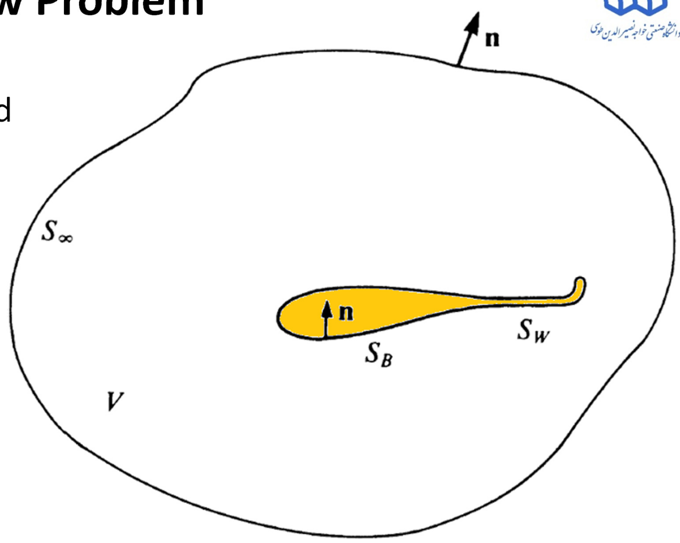
CHAPTER 3

General Solution of the Incompressible, Potential Flow Equations

Developing the basic methodology for obtaining the elementary solutions to potential flow problem.
Linear nature of the potential flow problem, the differential equation does not have to be solved individually for flow fields having different geometry at their boundaries.
Instead, the elementary solutions will be distributed in a manner that will satisfy each individual set of geometrical boundary conditions.

3.1 Statement of the Potential Flow Problem



The continuity equation for incompressible and irrotational

$$\nabla^2 \Phi = 0 \qquad (3.1)$$

The velocity component normal to the body's surface and to other solid boundaries must be zero, and in a body-fixed coordinate system

$$\nabla \Phi \cdot \mathbf{n} = 0 \qquad (3.2)$$

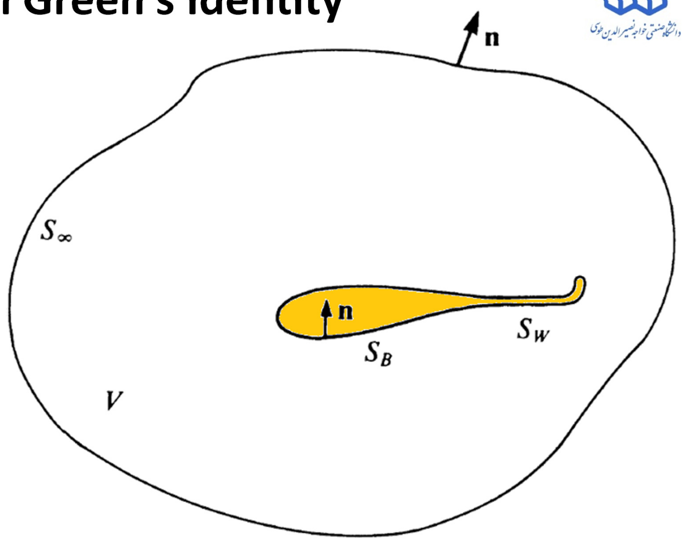
$\nabla\Phi$ is measured in a frame of reference attached to the body.
The disturbance created by the motion should decay far ($r \to \infty$) from the body

$$\lim_{r\to\infty} (\nabla \Phi - \mathbf{v}) = 0 \qquad (3.3)$$

where **r** = (x, y, z) and **v** is the relative velocity between the undisturbed fluid in V and the body (or the velocity at infinity seen by an observer moving with the body).

3.2 The General Solution, Based on Green's Identity

Solving Laplace's equation for the velocity potential for an arbitrary body with one of Green's identities



The divergence theorem Eq. (1.20)

$$\int_{c.s.} \mathbf{n}\cdot\mathbf{q}\,dS = \int_{c.v.} \nabla\cdot\mathbf{q}\,dV$$

q Replace by → $\Phi_1\nabla\Phi_2 - \Phi_2\nabla\Phi_1$

$$\int_S (\Phi_1\nabla\Phi_2 - \Phi_2\nabla\Phi_1)\cdot\mathbf{n}\,dS = \int_V (\Phi_1\nabla^2\Phi_2 - \Phi_2\nabla^2\Phi_1)\,dV \quad (3.4)$$

one of Green's identities

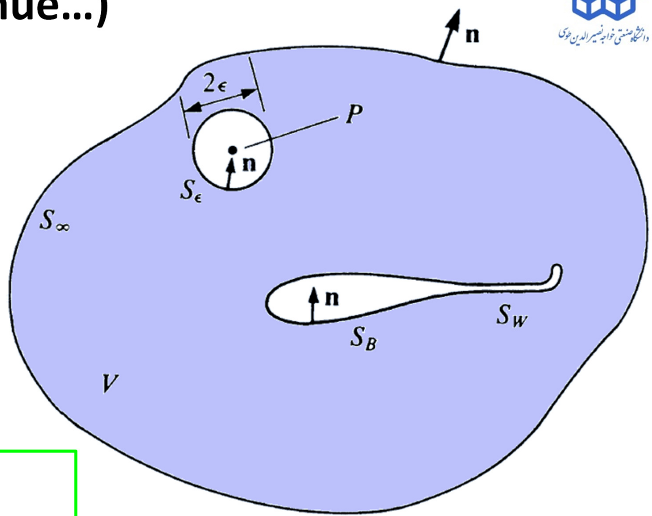
where Φ_1 and Φ_2 are two scalar functions of position

Wake Model

Volume integral → Surface integral → $S = S_B + S_W + S_\infty$

3.2 The General Solution, 3D (Continue…)

Let us set: $\Phi_1 = \dfrac{1}{r}$ and $\Phi_2 = \Phi$

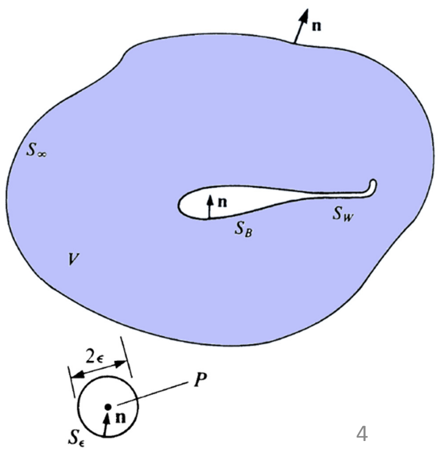


Φ: potential of the flow of interest in *V*

r : distance from a point *P*(*x, y, z*)

- *Case I* : Point *P* is outside of *V*
- *Case II* : Point *P* is inside V
- *Case III*: Point *P* lies on the boundary (for Example S_B)

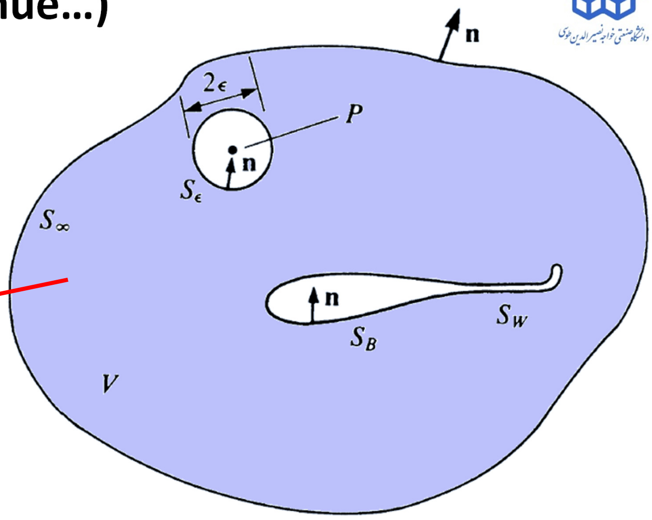
Case I : Point *P* is outside of *V*



Φ_1 and Φ_2 satisfy Laplace's equation in *V*

Eq. (3.4) → $$\int_S \left(\frac{1}{r}\nabla\Phi - \Phi\nabla\frac{1}{r}\right)\cdot\mathbf{n}\,dS = 0 \quad (3.6)$$

3.2 The General Solution, 3D (Continue…)



Case II : Point *P* is inside the region V

Φ_1 and Φ_2 satisfy Laplace's equation in V with excluded small sphere of radius ϵ

$$\begin{cases} \nabla^2(1/r) = 0 \\ \nabla^2\Phi_2 = 0 \end{cases}$$

Eq. (3.4) → $$\int_{S+\text{sphere } \epsilon}\left(\frac{1}{r}\nabla\Phi - \Phi\nabla\frac{1}{r}\right)\cdot\mathbf{n}\,dS = 0 \quad (3.6a)$$

Spherical coordinate system at *P*

$$\begin{cases} \mathbf{n} = -\mathbf{e}_r \\ \mathbf{n}\cdot\nabla\Phi = -\partial\Phi/\partial r \\ \nabla 1/r = -(1/r^2)\mathbf{e}_r \end{cases}$$

$$-\int_{\text{sphere } \epsilon}\left(\frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{\Phi}{r^2}\right)dS + \int_S\left(\frac{1}{r}\nabla\Phi - \Phi\nabla\frac{1}{r}\right)\cdot\mathbf{n}\,dS = 0 \quad (3.6b)$$

3.2 The General Solution, 3D (Continue…)

$$-\int_{\text{sphere } \epsilon}\left(\frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{\Phi}{r^2}\right)dS + \int_S\left(\frac{1}{r}\nabla\Phi - \Phi\nabla\frac{1}{r}\right)\cdot\mathbf{n}\,dS = 0 \quad (3.6b)$$

$$\int dS = 4\pi\epsilon^2$$
$$(\text{where } r = \epsilon)$$

$$\epsilon \to 0$$

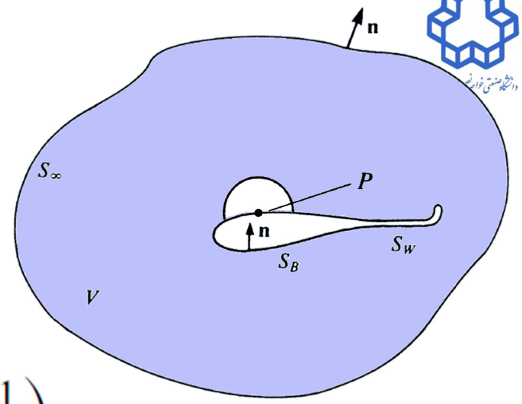
Potential and its derivatives are well-behaved functions and therefore do not vary much in the small sphere

$$-\int_{\text{sphere } \epsilon}\left(\frac{\Phi}{r^2}\right)dS = -4\pi\Phi(P)$$

Eq. (3.6b) → $$\Phi(P) = \frac{1}{4\pi}\int_S\left(\frac{1}{r}\nabla\Phi - \Phi\nabla\frac{1}{r}\right)\cdot\mathbf{n}\,dS \quad (3.7)$$

This formula gives the value of *Φ(P)* at any point in the flow, within the region *V*, in terms of the values of *Φ* and *∂Φ /∂n* on the boundaries *S*

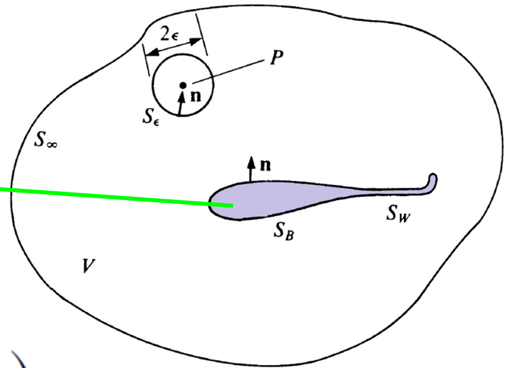
3.2 The General Solution, 3D (Continue…)



Case III: Point *P* lies on the boundary S_B

Integration around the hemisphere with radius ϵ

Eq. (3.7) → $$\Phi(P) = \frac{1}{2\pi}\int_S \left(\frac{1}{r}\nabla\Phi - \Phi\nabla\frac{1}{r}\right)\cdot \mathbf{n}\,dS \quad (3.7a)$$



Flow of interest occurs inside the boundary of S_B
Φ_i : internal potential
For this flow the point *P (which is in the region V)* is exterior to S_B

Eq. (3.6) → $$0 = \frac{1}{4\pi}\int_{S_B} \left(\frac{1}{r}\nabla\Phi_i - \Phi_i\nabla\frac{1}{r}\right)\cdot \mathbf{n}\,dS \quad (3.7b)$$

n points outward from S_B

3.2 The General Solution, 3D (Continue…)

Combination of Inner and outer Potential

Eq. (3.7) + Eq. (3.7b) →
$$\Phi(P) = \frac{1}{4\pi}\int_{S_B} \left[\frac{1}{r}\nabla(\Phi - \Phi_i) - (\Phi - \Phi_i)\nabla\frac{1}{r}\right]\cdot \mathbf{n}\,dS + \frac{1}{4\pi}\int_{S_W + S_\infty} \left(\frac{1}{r}\nabla\Phi - \Phi\nabla\frac{1}{r}\right)\cdot \mathbf{n}\,dS \quad (3.8)$$

The contribution of the S_∞

$$\Phi_\infty(P) = \frac{1}{4\pi}\int_{S_\infty} \left(\frac{1}{r}\nabla\Phi - \Phi\nabla\frac{1}{r}\right)\cdot \mathbf{n}\,dS \quad (3.9)$$

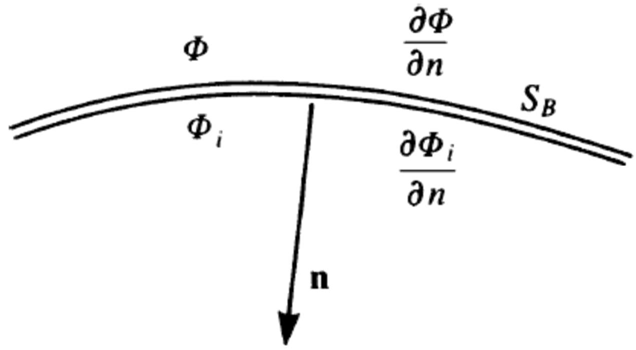
This potential depends on the selection of the coordinate sys. for example, in an inertial system where the body moves through an otherwise stationary fluid Φ_∞ =Constant

the wake surface is assumed to be thin, such that $\partial\Phi/\partial n$ is continuous across it (which means that no fluid-dynamic loads will be supported by the wake)

$$\Phi(P) = \frac{1}{4\pi}\int_{S_B} \left[\frac{1}{r}\nabla(\Phi - \Phi_i) - (\Phi - \Phi_i)\nabla\frac{1}{r}\right]\cdot \mathbf{n}\,dS - \frac{1}{4\pi}\int_{S_W} \Phi\mathbf{n}\cdot\nabla\frac{1}{r}dS + \Phi_\infty(P) \quad (3.10)$$

3.2 The General Solution, 3D (Continue…)

$\Phi(P)$ —— Reduced to Eq. (3.7) or Eq. (3.10) ——→ Determining values of Φ and $\partial\Phi/\partial n$ on the boundaries



Doublet: $-\mu = \Phi - \Phi_i \qquad (3.11)$

Source: $-\sigma = \dfrac{\partial \Phi}{\partial n} - \dfrac{\partial \Phi_i}{\partial n} \qquad (3.12)$

Eq. (3.10) →

$$\Phi(P) = -\frac{1}{4\pi}\int_{S_B}\left[\sigma\left(\frac{1}{r}\right) - \mu \mathbf{n}\cdot\nabla\left(\frac{1}{r}\right)\right]dS + \frac{1}{4\pi}\int_{S_W}\left[\mu \mathbf{n}\cdot\nabla\left(\frac{1}{r}\right)\right]dS + \Phi_\infty(P) \qquad (3.13)$$

Doublet strength μ is potential difference between the upper and lower wake surfaces (that is, if the wake thickness is zero, then $\mu = -\Delta\Phi$ on S_W)

3.2 The General Solution, 3D (Continue…)

Eq. (3.13) —— **n** .∇ by $\partial/\partial n$ ——→

$$\Phi(P) = -\frac{1}{4\pi}\int_{S_B}\left[\sigma\left(\frac{1}{r}\right) - \mu\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right]dS + \frac{1}{4\pi}\int_{S_W}\left[\mu\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right]dS + \Phi_\infty(P) \qquad (3.13a)$$

Source and doublet solutions decay as $r \to \infty$ and automatically fulfill the boundary condition of Eq. (3.3) (where **v** is the velocity due to Φ_∞)

Determining the strength of distribution of doublets and source on surface	→	To find the velocity potential in the region *V*

In Eq. (3.13) non-unique combination of sources and doublets for a particular problem
Choice based on the physics of the problem

$\dfrac{\partial \Phi_i}{\partial n} = \dfrac{\partial \Phi}{\partial n}$ on S_B → source term on S_B vanishes and only the doublet distribution remains

$\Phi_i = \Phi$ on S_B → doublet term on S_B vanishes

3.2 The General Solution, 2D (Continue…)

In the two-dimensional case

$$\Phi_1 = \ln r \qquad \text{and} \qquad \Phi_2 = \Phi \qquad (3.14)$$

Source Potential

Eq. (3.6b) → $$-\int_{\text{circle } \epsilon} \left(\ln r \frac{\partial \Phi}{\partial r} - \Phi \frac{1}{r} \right) dS + \int_S \left(\ln r \nabla \Phi - \Phi \nabla \ln r \right) \cdot \mathbf{n}\, dS = 0 \qquad (3.15)$$

Eq. (3.7) → $$\Phi(P) = -\frac{1}{2\pi} \int_S (\ln r \nabla \Phi - \Phi \nabla \ln r) \cdot \mathbf{n}\, dS \qquad (3.16)$$

If the point P lies on the boundary S_B

$$\Phi(P) = -\frac{1}{\pi} \int_S (\ln r \nabla \Phi - \Phi \nabla \ln r) \cdot \mathbf{n}\, dS \qquad (3.16a)$$

If the point P is inside S_B

Eq. (3.7b) → $$0 = -\frac{1}{2\pi} \int_{S_B} (\ln r \nabla \Phi_i - \Phi_i \nabla \ln r) \cdot \mathbf{n}\, dS \qquad (3.16b)$$

Eq. (3.13a) →

$$\Phi(P) = \frac{1}{2\pi} \int_{S_B} \left[\sigma \ln r - \mu \frac{\partial}{\partial n} (\ln r) \right] dS - \frac{1}{2\pi} \int_{S_W} \mu \frac{\partial}{\partial n} (\ln r)\, dS + \Phi_\infty(P) \qquad (3.17)$$

Note: $\partial/\partial n$ is the orientation of the doublet as will be illustrated in later and that the wake model S_W in the steady, two-dimensional lifting case is needed to represent a discontinuity in the potential Φ

3.3 Summary: Methodology of Solution

In 3D Eq. (3.13)
In 2D Eq. (3.17)
→ General approach to solution of incompressible potential flow problems

↓ $\nabla \Phi^2 = 0$

Distributing elementary solutions (sources and doublets) on the problem boundaries (S_B, S_W).

at $r \to \infty$ — Satisfy B.C. Eq. (3.3)

at r=0 — Velocity becomes singular, basic elements are called *singular solutions* →

↓ for general solution

Integration of basic solutions over any surface *S* containing these singularity elements

reduced to → finding appropriate singularity element distribution over some known boundaries

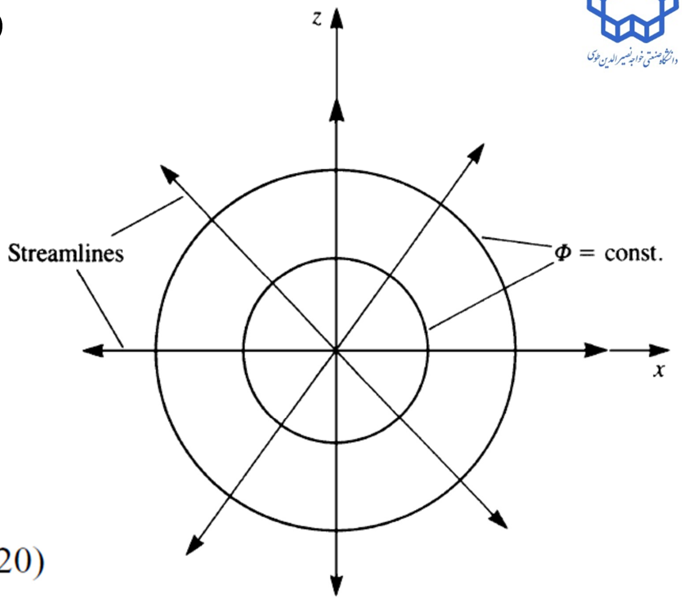
B.C. Eq. (3.2) will be fulfilled

$$\nabla \Phi = -\frac{1}{4\pi} \int_{S_B} \sigma \nabla \left(\frac{1}{r} \right) dS + \frac{1}{4\pi} \int_{S_B + S_W} \mu \nabla \left[\frac{\partial}{\partial n} \left(\frac{1}{r} \right) \right] dS + \nabla \Phi_\infty \qquad (3.18)$$

3.4 Basic Solution: Point Source 3D

One of the two basic solutions presented in Eq. (3.13)
Point source element placed at the origin of a spherical coordinate system

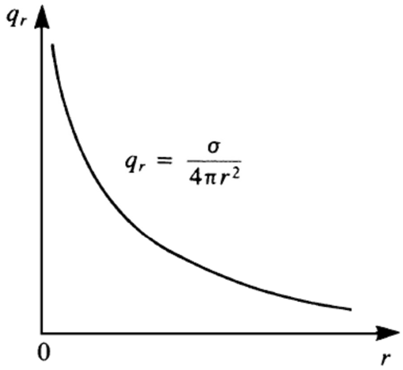
$$\Phi = -\frac{\sigma}{4\pi r} \qquad (3.19)$$



Velocity field with a radial component only

$$\mathbf{q} = -\frac{\sigma}{4\pi}\nabla\left(\frac{1}{r}\right) = \frac{\sigma}{4\pi}\frac{\mathbf{e}_r}{r^2} = \frac{\sigma}{4\pi}\frac{\mathbf{r}}{r^3} \qquad (3.20)$$

$$(q_r, q_\theta, q_\varphi) = \left(\frac{\partial \Phi}{\partial r}, 0, 0\right) = \left(\frac{\sigma}{4\pi r^2}, 0, 0\right) \qquad (3.21)$$

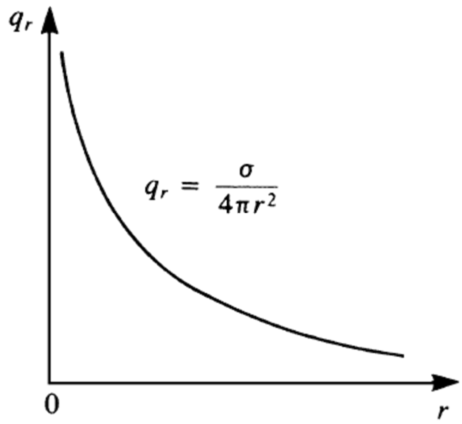


Velocity in the radial direction decays with the rate of $1/r^2$ and is singular at $r = 0$

3.4 Basic Solution: Point Source 3D (Continue)

The volumetric flow rate through a spherical surface of radius *r*

$$q_r 4\pi r^2 = \left(\frac{\sigma}{4\pi r^2}\right) \cdot 4\pi r^2 = \sigma$$



σ : Volumetric rate
- positive *σ :* *Source*
- negative *σ :* *Sink*

Note: this introduction of fluid at the source violates the conservation of mass; therefore, this point must be excluded from the region of solution

If the point element is located at a point $\mathbf{r_0}$

$$\Phi = \frac{-\sigma}{4\pi|\mathbf{r} - \mathbf{r}_0|} \qquad (3.22)$$

$$\mathbf{q} = \frac{\sigma}{4\pi}\frac{\mathbf{r} - \mathbf{r}_0}{|\mathbf{r} - \mathbf{r}_0|^3} \qquad (3.23)$$

3.4 Basic Solution: Point Source 3D (Continue)

The Cartesian form

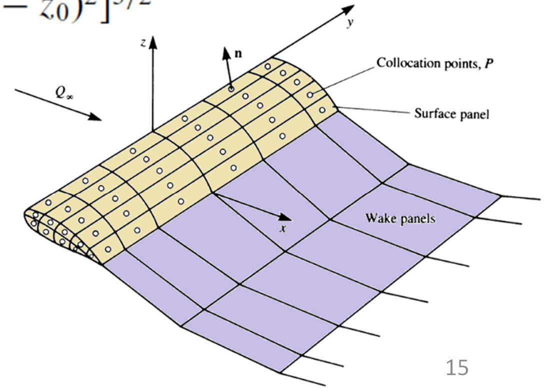
$$\Phi(x, y, z) = \frac{-\sigma}{4\pi\sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}} \qquad (3.24)$$

The velocity components

$$u(x, y, z) = \frac{\partial \Phi}{\partial x} = \frac{\sigma(x-x_0)}{4\pi[(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2]^{3/2}} \qquad (3.25a)$$

$$v(x, y, z) = \frac{\partial \Phi}{\partial y} = \frac{\sigma(y-y_0)}{4\pi[(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2]^{3/2}} \qquad (3.25b)$$

$$w(x, y, z) = \frac{\partial \Phi}{\partial z} = \frac{\sigma(z-z_0)}{4\pi[(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2]^{3/2}} \qquad (3.25c)$$



3.4 Basic Solution: Point Source 3D (Continue)

The basic point element (Eq. (3.24)) can be integrated over a line *l*, a surface *S*, or a volume *V* to create corresponding singularity elements

$$\Phi(x, y, z) = \frac{-1}{4\pi}\int_l \frac{\sigma(x_0, y_0, z_0)\, dl}{\sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}} \qquad (3.26)$$

$$\Phi(x, y, z) = \frac{-1}{4\pi}\int_S \frac{\sigma(x_0, y_0, z_0)\, dS}{\sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}} \qquad (3.27)$$

$$\Phi(x, y, z) = \frac{-1}{4\pi}\int_V \frac{\sigma(x_0, y_0, z_0)\, dV}{\sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}} \qquad (3.28)$$

Note: σ represents the source strength per unit length, area, and volume

The velocity components induced by these distributions can be obtained by differentiating the corresponding potentials

$$(u, v, w) = \left(\frac{\partial \Phi}{\partial x}, \frac{\partial \Phi}{\partial y}, \frac{\partial \Phi}{\partial z}\right)$$

3.5 Basic Solution: Point Doublet 3D

The second basic solution, presented in Eq. (3.13), is the doublet

$$\Phi = \frac{\mu}{4\pi}\mathbf{n}\cdot\nabla\left(\frac{1}{r}\right) \qquad (3.29)$$

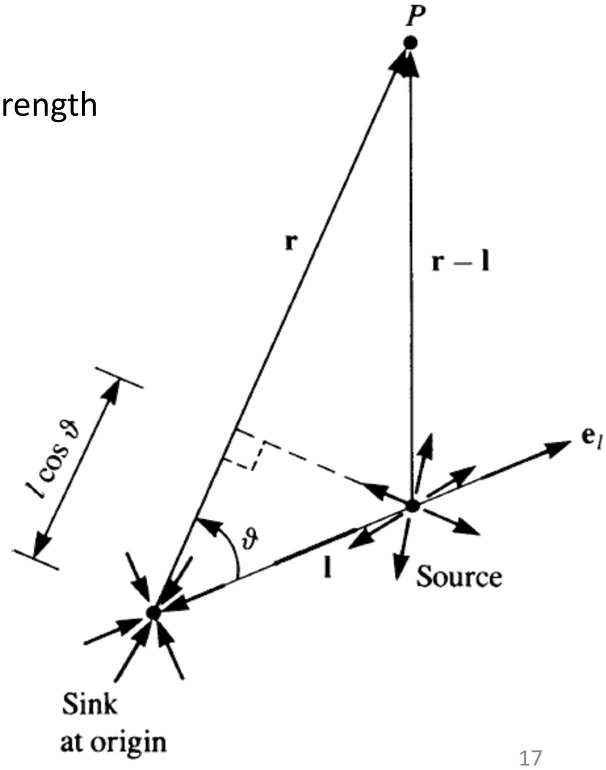
$$\Phi_{\text{doublet}} = -(\partial/\partial n)\Phi_{\text{source}}$$ for elements of unit strength

Developing Doublet element from Source element
a point sink at the origin and a point source at **l**

$$\Phi = \frac{\sigma}{4\pi}\left(\frac{1}{|\mathbf{r}|} - \frac{1}{|\mathbf{r}-\mathbf{l}|}\right) \qquad (3.30)$$

$l \to 0$ & $\sigma \to \infty$ such that $l\sigma \to \mu$ (μ is finite)

$$\Phi = \lim_{\substack{l\to 0\\ \sigma\to\infty\\ \sigma l\to\mu}} \frac{\sigma}{4\pi}\left(\frac{|\mathbf{r}-\mathbf{l}| - |\mathbf{r}|}{|\mathbf{r}||\mathbf{r}-\mathbf{l}|}\right)$$



3.5 Basic Solution: Point Doublet 3D (Continue)

As $l \to 0$
$$\begin{cases} |\mathbf{r}||\mathbf{r}-\mathbf{l}| \to r^2 \\ (|\mathbf{r}-\mathbf{l}| - |\mathbf{r}|) \to -l\ \cos\vartheta \end{cases}$$

Eq. (3.30) →
$$\Phi = \frac{-\mu}{4\pi}\frac{\cos\vartheta}{r^2} \qquad (3.31)$$

The angle ϑ is between the unit vector $\mathbf{e}_l$ pointing in the sink-to-source direction (doublet axis) and the vector **r**

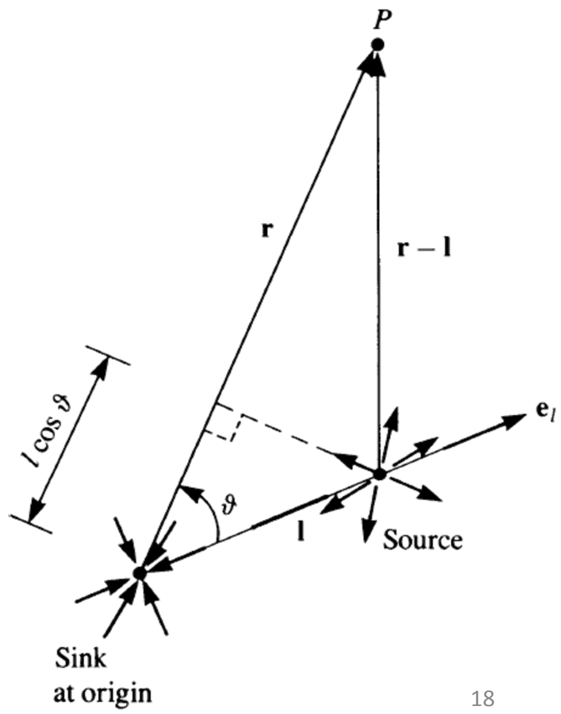
$\boldsymbol{\mu} = \mu\,\mathbf{e}_l$ Defining vector doublet strength

Eq. (3.31) →
$$\Phi = \frac{-\boldsymbol{\mu}\cdot\mathbf{r}}{4\pi r^3} \qquad (3.32)$$

if $\mathbf{e}_l$ in **n** direction

$$\Phi(P) = -\frac{1}{4\pi}\int_{S_B}\left[\sigma\left(\frac{1}{r}\right) - \mu\mathbf{n}\cdot\nabla\left(\frac{1}{r}\right)\right]dS + \frac{1}{4\pi}\int_{S_W}\left[\mu\mathbf{n}\cdot\nabla\left(\frac{1}{r}\right)\right]dS + \Phi_\infty(P) \qquad (3.13)$$

$$\Phi = \frac{\mu}{4\pi}\mathbf{n}\cdot\nabla\left(\frac{1}{r}\right) \qquad (3.29)$$



3.5 Basic Solution: Point Doublet 3D (Continue)

$$\Phi_{\text{doublet}} = \frac{-\mathbf{e}_l \cdot \mathbf{r}}{4\pi r^3} = -\mathbf{e}_l \cdot \nabla\left(\frac{-1}{4\pi r}\right) = -\frac{\partial}{\partial n}\Phi_{\text{source}} \qquad (3.33)$$

For example, for a doublet at the origin and the doublet strength vector (μ, 0, 0) aligned with the x axis ($\mathbf{e}_l = \mathbf{e}_x$ and $\vartheta = \theta$), the potential in spherical coordinates is

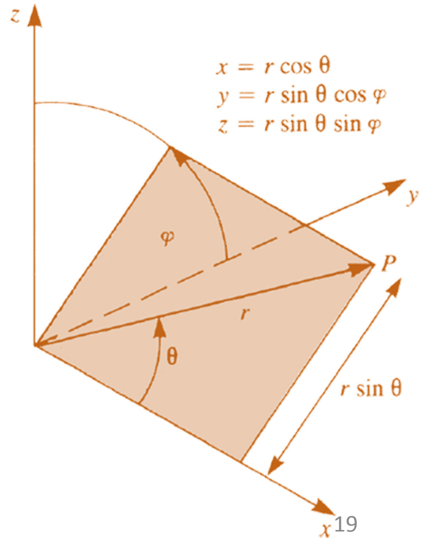
$$\Phi(r, \theta, \varphi) = \frac{-\mu \cos\theta}{4\pi r^2} \qquad (3.34)$$

The velocity potential due to such doublet elements, located at (x_0, y_0, z_0), is

$$\Phi(x, y, z) = \frac{\mu}{4\pi} n \cdot \nabla\left(\frac{1}{|\mathbf{r} - \mathbf{r}_0|}\right) = \frac{\mu}{4\pi}\frac{\partial}{\partial n}\left(\frac{1}{|\mathbf{r} - \mathbf{r}_0|}\right) \qquad (3.35)$$

$\partial/\partial n$ as the derivative in the direction of the three axes

$$\Phi(x, y, z) = \frac{\mu}{4\pi}\begin{pmatrix}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial y}\\ \frac{\partial}{\partial z}\end{pmatrix}\frac{1}{\sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}} \qquad (3.36)$$



3.5 Basic Solution: Point Doublet 3D (Continue)

Equation (3.34) shows that the doublet element does not have a radial symmetry but rather has a directional property. Therefore, in Cartesian coordinates three elements are defined

Eq. (3.36) →

$$\Phi(x, y, z) = \frac{-\mu}{4\pi}(x - x_0)[(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2]^{-3/2} \qquad (3.37)$$

$$\Phi(x, y, z) = \frac{-\mu}{4\pi}(y - y_0)[(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2]^{-3/2} \qquad (3.38)$$

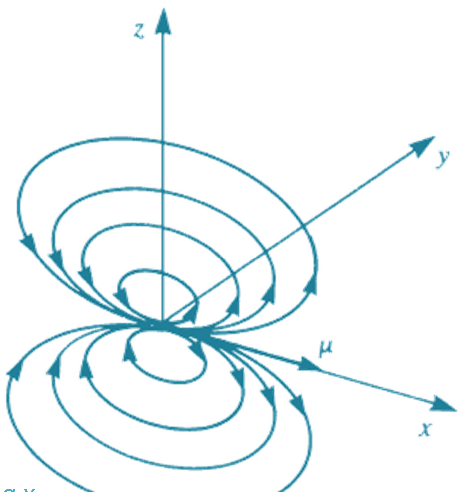
$$\Phi(x, y, z) = \frac{-\mu}{4\pi}(z - z_0)[(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2]^{-3/2} \qquad (3.39)$$

The velocity components in spherical coordinates for x- directional point doublet (μ, 0, 0)

$$q_r = \frac{\partial\Phi}{\partial r} = \frac{\mu\cos\theta}{2\pi r^3} \qquad (3.40)$$

$$q_\theta = \frac{1}{r}\frac{\partial\Phi}{\partial\theta} = \frac{\mu\sin\theta}{4\pi r^3} \qquad (3.41)$$

$$q_\varphi = \frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\varphi} = 0 \qquad (3.42)$$



velocity field, due to a x directional point doublet (μ, 0, 0)

3.5 Basic Solution: Point Doublet 3D (Continue)

The velocity components in Cartesian coordinates for *x*- directional point doublet (μ, 0, 0)

Differentiating Eq. (3.37) →

$$u = -\frac{\mu}{4\pi}\frac{(y-y_0)^2+(z-z_0)^2-2(x-x_0)^2}{[(x-x_0)^2+(y-y_0)^2+(z-z_0)^2]^{5/2}} \quad (3.43)$$

$$v = \frac{3\mu}{4\pi}\frac{(x-x_0)(y-y_0)}{[(x-x_0)^2+(y-y_0)^2+(z-z_0)^2]^{5/2}} \quad (3.44)$$

$$w = \frac{3\mu}{4\pi}\frac{(x-x_0)(z-z_0)}{[(x-x_0)^2+(y-y_0)^2+(z-z_0)^2]^{5/2}} \quad (3.45)$$

This basic point element can be integrated over a line *l*, a surface *S*, or a volume *V* to create the corresponding singularity elements (for (μ, 0, 0))

$$\Phi(x,y,z) = \frac{-1}{4\pi}\int_l \frac{\mu(x_0,y_0,z_0)\cdot(x-x_0)\,dl}{[(x-x_0)^2+(y-y_0)^2+(z-z_0)^2]^{3/2}} \quad (3.46)$$

$$\Phi(x,y,z) = \frac{-1}{4\pi}\int_S \frac{\mu(x_0,y_0,z_0)\cdot(x-x_0)\,dS}{[(x-x_0)^2+(y-y_0)^2+(z-z_0)^2]^{3/2}} \quad (3.47)$$

$$\Phi(x,y,z) = \frac{-1}{4\pi}\int_V \frac{\mu(x_0,y_0,z_0)\cdot(x-x_0)\,dV}{[(x-x_0)^2+(y-y_0)^2+(z-z_0)^2]^{3/2}} \quad (3.48)$$

3.6 Basic Solution: Polynomials 3D

Laplace's equation is a 2nd order PDE — linear function can be be a solution →

$$\Phi = Ax + By + Cz \quad (3.49)$$

velocity components

$$u = \frac{\partial\Phi}{\partial x} = A \equiv U_\infty, \qquad v = \frac{\partial\Phi}{\partial y} = B \equiv V_\infty, \qquad w = \frac{\partial\Phi}{\partial z} = C \equiv W_\infty \quad (3.50)$$

Where U_∞, V_∞, and W_∞ are constant velocity components in the x, y, and z directions.
the velocity potential due a constant free-stream flow in the *x direction is*

$$\Phi = U_\infty x \quad (3.51)$$

and in general

$$\Phi = U_\infty x + V_\infty y + W_\infty z \quad (3.52)$$

3.6 Basic Solution: Polynomials 3D (Continue)

Additional polynomial solutions can be sought

$$\Phi = Ax^2 + By^2 + Cz^2 \quad (3.53) \xrightarrow[\text{Laplace's Eq}]{\text{to satisfy}} \nabla^2\Phi = A + B + C = 0$$

A, B, and *C* should be constants. There are numerous combinations of constants that will satisfy this condition.

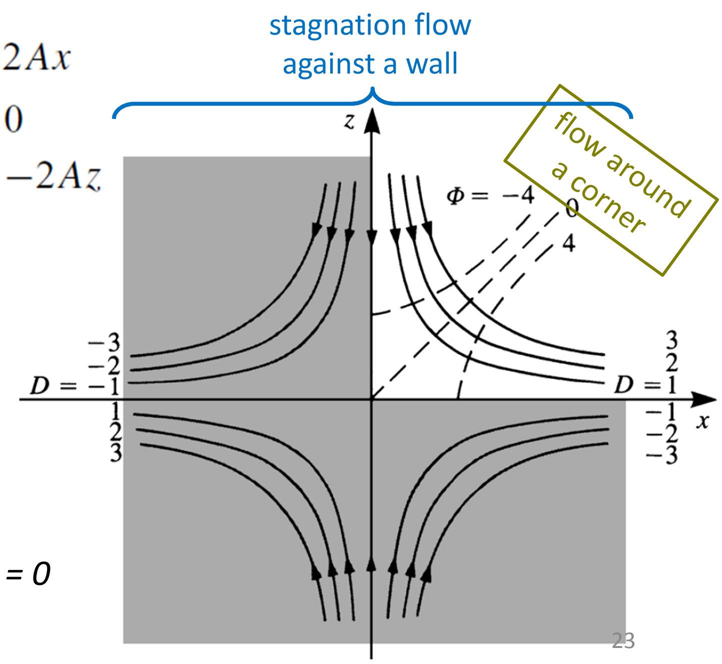
$$A = -C \rightarrow \Phi = A(x^2 - z^2) \rightarrow \begin{cases} u = 2Ax \\ v = 0 \\ w = -2Az \end{cases}$$

Streamline equation (Eq. 1.6a)

$$\frac{dx}{u} = \frac{dz}{w} \rightarrow \frac{dx}{2Ax} = \frac{dz}{-2Az}$$

$$xz = \text{const.} = D$$

At origin *x= z = 0,* velocity components *u = w = 0*
stagnation point



3.7 2D Version of the Basic Solutions (Source)

At 3D source element $\begin{cases} q_r \neq 0 \\ q_\theta = 0 \\ q_\varphi = 0 \end{cases}$ → At 2D source element $\begin{cases} q_r \neq 0 \\ q_\theta = 0 \end{cases}$

For Irrotational flow →
$$\zeta_y = 2\omega_y = -\frac{1}{r}\left[\frac{\partial}{\partial r}(rq_\theta) - \frac{\partial}{\partial \theta}(q_r)\right] = \frac{1}{r}\frac{\partial}{\partial \theta}(q_r) = 0$$

q_r function of r only
$q_r = q_r(r)$

Satisfing the continuity equation (Eq. (1.35))

$$\nabla \cdot \mathbf{q} = \frac{dq_r}{dr} + \frac{q_r}{r} = \frac{1}{r}\frac{d}{dr}(rq_r) = 0 \rightarrow rq_r = \text{const.} = \sigma/2\pi$$

σ: area flow rate passing across a circle of radius *r.*
velocity components for a source element at the origin:

$$q_r = \frac{\partial \Phi}{\partial r} = \frac{\sigma}{2\pi r} \qquad (3.57)$$

$$q_\theta = \frac{1}{r}\frac{\partial \Phi}{\partial \theta} = 0 \qquad (3.58)$$

Integrating →
$$\Phi = \frac{\sigma}{2\pi}\ln r + C \qquad (3.59)$$

Velocity Potential
C Can be set zero

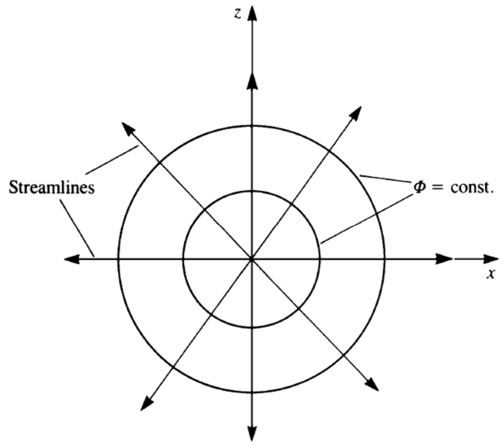
r = 0 is a singular point and must be excluded from region of solution

$$q_r\, 2\pi R = \frac{\sigma}{2\pi R} 2\pi R = \sigma$$

σ: strength of the source

3.7 2D Version of the Basic Solutions (Source)

In Cartesian coordinates with source located at (x_0, z_0)



$$\Phi(x,z) = \frac{\sigma}{2\pi} \ln \sqrt{(x-x_0)^2 + (z-z_0)^2} \qquad (3.60)$$

$$u = \frac{\partial \Phi}{\partial x} = \frac{\sigma}{2\pi} \frac{x - x_0}{(x-x_0)^2 + (z-z_0)^2} \qquad (3.61)$$

$$w = \frac{\partial \Phi}{\partial z} = \frac{\sigma}{2\pi} \frac{z - z_0}{(x-x_0)^2 + (z-z_0)^2} \qquad (3.62)$$

In 2D stream fuction Eqs. (2.80a,b)

$$q_\theta = -\frac{\partial \Psi}{\partial r} = 0 \qquad (3.63)$$

$$q_r = \frac{1}{r}\frac{\partial \Psi}{\partial \theta} = \frac{\sigma}{2\pi r} \qquad (3.64)$$

Integrating → Integr. const.= zero

$$\Psi = \frac{\sigma}{2\pi}\theta \qquad (3.65)$$

Velocity Potential $\Phi = \frac{\sigma}{2\pi} \ln r$

$\Psi = \frac{\sigma}{2\pi}\theta$ Stream Function

3.7 2D Version of the Basic Solutions (Doulet)

2D doublet can be obtained by a point source and a point sink approach each other

3D Doublet Eq. (3.32) → $\Phi(r) = \frac{-\boldsymbol{\mu} \cdot \mathbf{r}}{2\pi r^2}$ (3.66)

$\Phi_{\text{doublet}} = -\frac{\partial}{\partial n}\Phi_{\text{source}}$ → $\Phi(r) = -\frac{\partial}{\partial n}\frac{\sigma}{2\pi} \ln r$ (3.67)

Eq. (3.33)

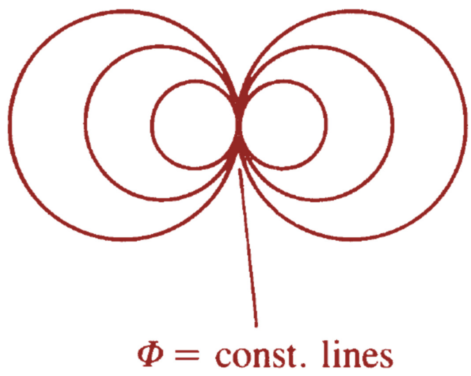
Replacing the source strength by μ & with $\boldsymbol{n}$ in the x direction $\boldsymbol{\mu} = (\mu, 0)$

Eq. (3.66) → $\Phi(r,\theta) = \frac{-\mu}{2\pi}\frac{\cos\theta}{r}$ (3.68)

The Velocity field by differentiating the velocity potential

$$q_r = \frac{\partial \Phi}{\partial r} = \frac{\mu \cos\theta}{2\pi r^2} \qquad (3.69)$$

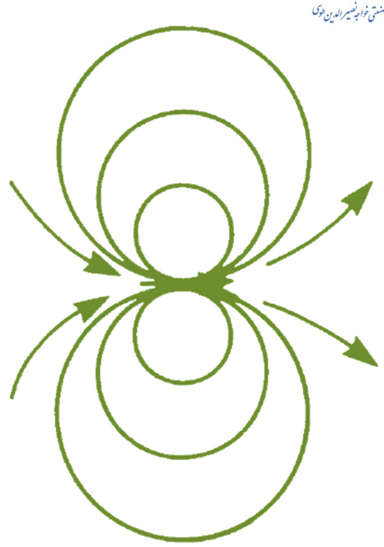
$$q_\theta = \frac{1}{r}\frac{\partial \Phi}{\partial \theta} = \frac{\mu \sin\theta}{2\pi r^2} \qquad (3.70)$$



3.7 2D Version of the Basic Solutions (Doulet)

The velocity potential in Cartesian, doublet at the point (x_0, z_0)

$$\Phi(x,z) = \frac{-\mu}{2\pi}\frac{x-x_0}{(x-x_0)^2+(z-z_0)^2} \quad (3.71)$$



The velocity components

$$u = \frac{\mu}{2\pi}\frac{(x-x_0)^2-(z-z_0)^2}{[(x-x_0)^2+(z-z_0)^2]^2} \quad (3.72)$$

$$w = \frac{\mu}{2\pi}\frac{2(x-x_0)(z-z_0)}{[(x-x_0)^2+(z-z_0)^2]^2} \quad (3.73)$$

Deriving the stream function for this doublet element

$$q_\theta = -\frac{\partial \Psi}{\partial r} = \frac{\mu \sin\theta}{2\pi r^2} \quad (3.74)$$

$$q_r = \frac{1}{r}\frac{\partial \Psi}{\partial \theta} = \frac{\mu \cos\theta}{2\pi r^2} \quad (3.75)$$

Integrating → Integr. const.= zero

$$\Psi = \frac{\mu \sin\theta}{2\pi r} \quad (3.76)$$

Note: a similar doublet element where $\boldsymbol{\mu}$ = (0,μ) can be derived by using Eq. (3.66)

3.8 Basic Solution: Vortex

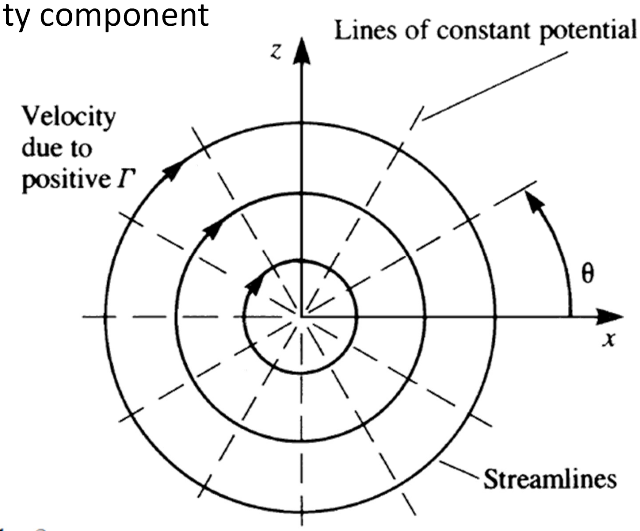
- In 2D Point Vortex → Velocity Potential & Velocity Field
- In 3D Vortex Filament → Velocity Field by Biot–Savart Law

A singularity element with only a tangential velocity component
Velocity components:

$$q_r = 0$$

$$q_\theta = q_\theta(r,\theta)$$

Substitution in continuity equation (Eq. (1.35)) → $q_\theta = q_\theta(r)$



For irrotational flow

$$\zeta_y = 2\omega_y = -\frac{1}{r}\left[\frac{\partial}{\partial r}(rq_\theta) - \frac{\partial}{\partial \theta}(q_r)\right] = -\frac{1}{r}\frac{\partial}{\partial r}(rq_\theta) = 0$$

Integrating with respect to r

$$rq_\theta = \text{const.} = A$$

3.8 Basic Solution: Vortex (Continue)

Calculating *A* by using the definition of the circulation

$$\Gamma = \oint \mathbf{q} \cdot d\mathbf{l} = \int_{2\pi}^{0} q_\theta \cdot r\, d\theta = -2\pi A \longrightarrow A = -\frac{\Gamma}{2\pi}$$

Note: Γ is positive in clockwise

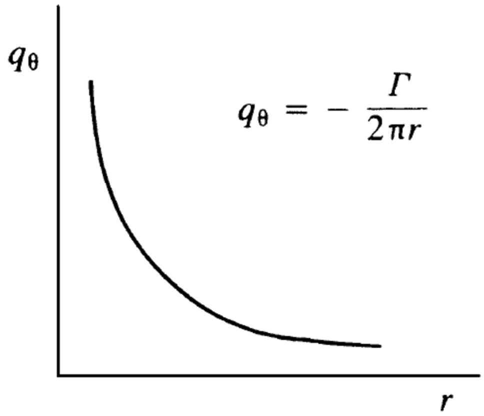
The velocity field:

$$q_r = 0 \qquad (3.77)$$

$$q_\theta = -\frac{\Gamma}{2\pi r} \qquad (3.78)$$

The tangential velocity component decays at a rate of $1/r$

The velocity potential for a vortex element at the origin



integration →

$$\Phi = \int q_\theta r\, d\theta + C = -\frac{\Gamma}{2\pi}\theta + \cancel{C} \qquad (3.79)$$

Integrating around a vortex we do find vorticity concentrated at a zero-area point, but with finite circulation. if we integrate ***q . dl*** around any closed curve in the field (not surrounding the vortex) the value of the integral will be zero.

The vortex is a solution to the Laplace equation and results in an irrotational flow, excluding the vortex point itself.

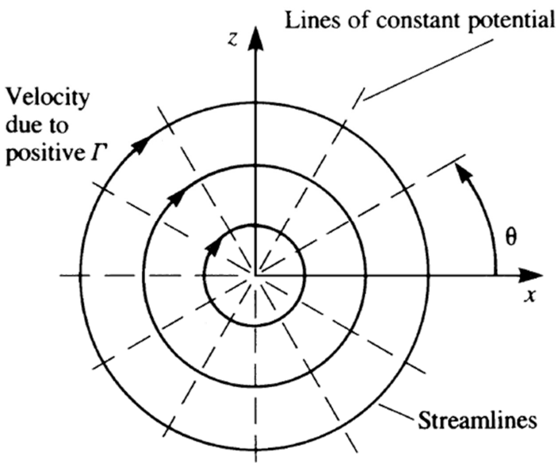
3.8 Basic Solution: Vortex (Continue)

In Cartesian coordinates for a vortex located at (x_0, z_0)

$$\Phi = -\frac{\Gamma}{2\pi}\tan^{-1}\frac{z - z_0}{x - x_0} \qquad (3.80)$$

$$u = \frac{\Gamma}{2\pi}\frac{z - z_0}{(z - z_0)^2 + (x - x_0)^2} \qquad (3.81)$$

$$w = -\frac{\Gamma}{2\pi}\frac{x - x_0}{(z - z_0)^2 + (x - x_0)^2} \qquad (3.82)$$



Deriving stream function for 2D vortex located at the origin, in *x–z* or (r– θ) plane

$$q_\theta = -\frac{\partial \Psi}{\partial r} = -\frac{\Gamma}{2\pi r} \qquad (3.83)$$

$$q_r = \frac{1}{r}\frac{\partial \Psi}{\partial \theta} = 0 \qquad (3.84)$$

Integrating → Integr. const.= zero

$$\Psi = \frac{\Gamma}{2\pi}\ln r \qquad (3.85)$$

The streamlines where *Ψ= const*

3.9 Principle of Superposition

If $\varphi_1, \varphi_2, \ldots, \varphi_n$ are solutions of the Laplace equation, which is linear, then

$$\Phi = \sum_{k=1}^{n} c_k \Phi_k \qquad (3.86)$$

→ a solution for Laplace equation in that region

$$\nabla^2 \Phi = \sum_{k=1}^{n} c_k \nabla^2 \Phi_k = 0$$

Where $c_1, c_2, \ldots, c_n$ are arbitrary constants

This superposition principle is a very important property of the Laplace equation, paving the way for solutions of the flowfield near complex boundaries. In theory, by using a set of elementary solutions, the solution process (of satisfying a set of given boundary conditions) can be reduced to an algebraic search for the right linear combination of these elementary solutions.